

Functional renormalisation group approach to far-from-equilibrium quantum field dynamics

Thomas Gasenzer* and Jan M. Pawłowski†

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany

Dynamic equations for quantum fields far from equilibrium are derived by use of functional renormalisation group techniques. The obtained equations are non-perturbative and lead substantially beyond mean-field and quantum Boltzmann type approximations. The approach is based on a regularised version of the generating functional for correlation functions where times greater than a chosen cutoff time are suppressed. As a central result, a time evolution equation for the non-equilibrium effective action is derived, and the time evolution of the Green functions is computed within a vertex expansion. It is shown that this agrees with the dynamics derived from the $1/N$ -expansion of the two-particle irreducible effective action.

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Introduction. Far-from-equilibrium quantum field dynamics is one of the most challenging issues both in experimental and theoretical physics to date. Experiments exhibiting quantum statistical effects in the time evolution of many-body systems are extremely demanding. In particular, the preparation of ultracold atomic Bose and Fermi gases in various trapping environments allows to precisely study quantum many-body dynamics of strongly correlated systems, see, e.g., Refs. [1]. In recent years, the field has attracted researchers from a variety of disciplines, ranging from condensed-matter to high-energy particle physics and cosmology. Nonequilibrium field theory to date is dominated by semi-classical mean-field approaches which are in general only valid for weak interactions or large occupation numbers. For strongly correlated quantum systems methods are available predominantly for systems in one spatial dimension and include the Density Matrix Renormalisation Group methods, see e.g. [2], as well as techniques for exactly solvable models [3]. Non-perturbative approximations of the two-particle irreducible (2PI) effective action [4] have been intensively studied and applied to nonequilibrium dynamics [5, 6, 7, 8, 9, 10, 11] and are applicable also in more than one dimension. In field theory, they also provide a way to study strongly correlated fermions beyond mean-field and Boltzmann approximations [8]. Like these, the results presented here are expected to be of high relevance, e.g., for the description of ultracold degenerate Fermi gases close to the BEC-BCS crossover [12].

In this paper we derive dynamic equations for quantum fields far from equilibrium. As a central result, we derive a new time evolution equation for the non-equilibrium effective action by use of functional renormalisation group (RG) techniques, cf. Refs. [13, 14, 15, 16], as well as [17] for non-equilibrium applications. This exact and closed evolution equation allows for non-perturbative approximations that lead substantially beyond mean-field and quantum Boltzmann type approaches: it can be rewritten, in a closed form, as a hierarchy of dynamical equations for Green functions, and this hierarchy admits truncations that neither explicitly nor implicitly rely on small bare couplings or close-to-equilibrium evolutions. Moreover, the hierarchy of equations is technically very close to 2PI and Dyson-Schwinger type evolutions, see [16], as well as to evolution equations for the effective action

as derived in [18]. This has the great benefit that results from either method can be used as an input as well as for reliability checks of the respective truncation schemes. In turn, in particular the different resummation schemes invoked in these approaches, as well as the differing dependences on Green functions, allow for a systematic analysis of the mechanisms of non-equilibrium physics. Here we apply the above setting within a non-perturbative vertex expansion scheme as well as an s -channel approximation. This truncation turns out to correspond to an expansion in inverse powers of the number of field components \mathcal{N} . The dynamic equations derived agree to next-to-leading order with those obtained from a $1/\mathcal{N}$ expansion of the 2PI effective action.

Functional renormalisation group approach. For a given initial-state density matrix $\rho_D(t_0)$, the renormalised finite quantum generating functional for time-dependent n -point correlation functions,

$$Z[J; \rho_D] = \text{Tr} \left[\rho_D(t_0) \mathcal{T}_C \exp \left\{ i \int_{x,C} J_a(x) \Phi_a(x) \right\} \right], \quad (1)$$

(summation over double indices is implied) carries all the information of the quantum many-body evolution at times greater than the initial time t_0 . The Heisenberg field operators $\Phi_a(x)$, $a = 1, \dots, \mathcal{N}$, are assumed to obey equal-time bosonic commutation relations. In Eq. (1), \mathcal{T}_C denotes time-ordering along the Schwinger-Keldysh closed time path (CTP) \mathcal{C} leading from t_0 along the real-time axis to infinity and back to t_0 , with $\int_{x,C} \equiv \int_C dx_0 \int d^d x$. All connected Greens functions will be time-ordered along \mathcal{C} .

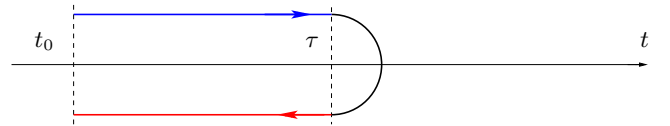


FIG. 1: (color online) The closed time path $\mathcal{C}(\tau)$ terminating at the time given by the parameter τ . At later times, field fluctuations summed over in the generating functional do not contribute to Green functions the maximum time argument of which is τ .

The key idea of our approach to dynamics is to first consider the generating functional for Green functions where all times

are smaller than a maximum time τ . This implies a time path $\mathcal{C}(\tau)$ which is closed at $t = \tau$, see Fig. 1, and we are led to the generating functional $Z_\tau = Z_{\mathcal{C}(\tau)}$ with the source term

$$\mathcal{T}_{\mathcal{C}(\tau)} \exp \left\{ i \int_{x, \mathcal{C}(\tau)} J_a(x) \Phi_a(x) \right\}. \quad (2)$$

At $\tau = t_0$, this results in a trivial Z_{t_0} where all information is stored in the initial density matrix $\rho_D(t_0)$. From this initial condition Z_τ can be computed by means of the time evolution $\partial_\tau Z_\tau$ for all times $\tau > t_0$.

We will derive this evolution by using functional RG ideas. To that end we note that Z_τ can be defined in terms of the full generating functional Z_∞ in (1) by suppressing the propagation for times greater than τ . This suppression is achieved by

$$Z_\tau = \exp \left\{ - \frac{i}{2} \int_{xy, \mathcal{C}} \frac{\delta}{\delta J_a(x)} R_{\tau, ab}(x, y) \frac{\delta}{\delta J_b(y)} \right\} Z, \quad (3)$$

where the function R_τ is chosen such that it suppresses the fields, i.e., $\delta/\delta J_a$, for all times $t > \tau$. This requirement does not fix R_τ in a unique way, and a simple choice is provided by

$$-iR_{\tau, ab}(x, y) = \begin{cases} \infty & \text{for } x_0 = y_0 > \tau, \mathbf{x} = \mathbf{y}, a = b \\ 0 & \text{otherwise} \end{cases}, \quad (4)$$

see Fig. 2. We note that for $\tau \rightarrow \infty$, we recover the full generating functional, $Z_\infty \equiv Z$, while for $\tau = t_0$, any evolution to times $t > t_0$ is suppressed. In the latter case all n -point functions derived from Z_τ are defined at t_0 only and reduce to the free classical ones, or any other physical boundary condition.

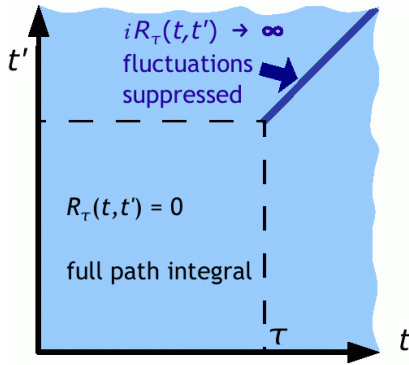


FIG. 2: (Color online) The cutoff function $R_{\tau, ab}(x, y)$ in the time plane $\{x_0, y_0\} = \{t, t'\}$, $t, t' \geq t_0$. The function vanishes everywhere except for $t = t' > \tau$ where it tends to infinity and therefore implies a suppression of all fluctuations in the generating functional at times greater than τ .

We emphasise that the cutoff R_τ in (3) suppresses any time evolution at times greater than τ . Correlation functions derived from Z_τ vanish as soon as at least one of their time arguments is larger than τ . Hence, the regularised generating

functional (3) is equivalent to a generating functional with a closed time path $\mathcal{C}(\tau)$ leading from t_0 to τ and back to t_0 . Note that the CTP automatically arranges for the normalisation of Z_τ . We conclude that, by construction, the sharp cutoff (4) is a physical one in the sense that it corresponds to integrating out all fluctuations being relevant for the evolution up to a particular time.

The restriction of the CTP to times $t_0 \leq t \leq \tau$ implies that the differential equation for Z_τ describing the flow of the generating functional, and therefore that of the correlation functions, encodes the full time evolution of the system. Analogously, the time evolution of connected correlation functions is derived from that of the Schwinger functional $W_\tau = -i \ln Z_\tau$. It is more convenient, however, to work with the effective action,

$$\Gamma_\tau[\phi; R_\tau] = W_\tau[J; \rho_D] - \int_{\mathcal{C}} J_a \phi_a - \frac{1}{2} \int_{\mathcal{C}} \phi_a R_{\tau, ab} \phi_b. \quad (5)$$

Here, space-time arguments are suppressed, and $\phi_a(x) = \delta W_\tau / \delta J_a(x)|_{J \equiv 0}$ is the classical field expectation value. From Eqs. (3) and (5) we derive the Functional RG or flow equation for the τ -dependent effective action,

$$\partial_\tau \Gamma_\tau = \frac{i}{2} \int_{\mathcal{C}} \left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau} \right]_{ab} \partial_\tau R_{\tau, ab}, \quad (6)$$

where $\Gamma_\tau^{(n)} = \delta^n \Gamma_\tau / (\delta \phi)^n$. Again, space-time arguments are suppressed which appear in analogy to the field indices a, b , see e.g. Ref. [16]. Eq. (6) represents our central result. It is analogous to functional flow equations used extensively with regulators in momentum and/or frequency space to describe strongly correlated systems near equilibrium [13, 14, 15, 16]. Its homogenous part relates to standard τ -dependent renormalisation [16], and has been studied e.g. in [20, 21]. We close the derivation of the evolution equation with some remarks: Eq. (6) is nothing but an infinitesimal closed time loop at τ , which is by itself finite and requires no further renormalisation. However, in particular in higher dimensions additional regulators in the spatial or momentum domains may be advantageous for facilitating renormalisation and thus the practical application of the approach, see [13, 14, 16]. We also emphasise that the evolution equation (6), even though close in spirit and construction to standard functional RG equations, is conceptually different. It entails a physical time evolution as opposed to integrating out degrees of freedom. Nevertheless, more general regulators R_τ may be advantageous in other cases [16], e.g., when a cutoff is set in the relative time ($t - t'$) direction, see Fig. 2. Such a case would correspond to setting a cutoff along the frequency axis [17] which is conceptually closer to the standard RG approach with cutoffs in the momentum domain. A more detailed discussion of our approach is deferred to [19].

Flow equations for correlation functions. To obtain a practically solvable set of dynamic equations, we derive the flow equation for the proper n -point Green function $\Gamma_\tau^{(n)}$ by taking

the n th field derivative of Eq. (6). Fig. 3 shows a diagrammatic representation of the resulting equations for the τ -dependent proper two- and four-point functions.

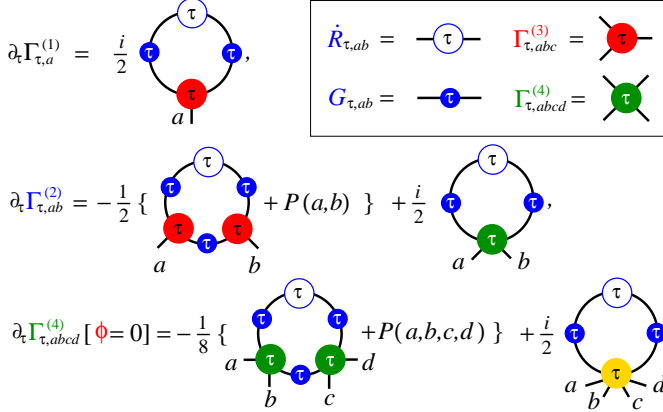


FIG. 3: (color online) Diagrammatic representation of the general flow equations for $\Gamma_\tau^{(1)}[\phi]$, $\Gamma_\tau^{(2)}[\phi]$, and $\Gamma_{\tau,abcd}^{(4)}[\phi=0]$, for a ϕ^4 -theory. Open circles with a τ denote $\partial_\tau R_{\tau,ab}$, solid lines with (blue) filled circles are τ - and, in general, ϕ -dependent two-point functions $G_{\tau,ab} = i[\Gamma_\tau^{(2)} + R_\tau]_{ab}^{-1}$. All other filled circles denote proper field-dependent n -vertices $\Gamma_{\tau,abcd}^{(n)}$, $n = 3, 4, 6$. P implies a sum corresponding to all permutations of its arguments.

We would like to emphasise that proper n -point Green functions in general do not vanish for times greater than τ as opposed to the correlation functions derived from Z_τ . However, in contrast to usual renormalisation-group flows in the momentum domain, causality prevents the influence of proper Green functions for times greater than τ on the dynamics up to time τ .

To be more specific, we consider in the following the special case of an \mathcal{N} -component scalar ϕ^4 theory defined by the classical action

$$S[\varphi] = \frac{1}{2} \int_{xy} \varphi_a(x) i G_{0,ab}^{-1}(x, y) \varphi_b(y) - \frac{g}{4\mathcal{N}} \int_x (\varphi(x)^2)^2, \quad (7)$$

where $\varphi^2 = \varphi_a \varphi_a$, and where the inverse free classical propagator involves one or more time derivatives, e.g., $G_{0,ab}^{-1}(x, y) = \delta_C(x - y) [-\sigma_{ab}^2 \partial_{x_0} + i H_{1B}(x) \delta_{ab}]$ for a non-relativistic trapped Bose gas, with σ^2 being the Pauli 2-matrix and $H_{1B}(x) = -\Delta_x^2/2m + V(x)$ the one-body Hamiltonian.

Our goal is to derive the full time-evolution of $\Gamma^{(n)} = \Gamma_\tau^{(n)}$, in particular, of the connected two-point function $G = i[\Gamma^{(2)}]^{-1} = i[\Gamma_\tau^{(2)}]^{-1}$ which contains all information about the normal and anomalous one-body density matrices (cf., e.g. Ref. [9]). As a consequence of the above mentioned effective cut off of the CTP at times greater than τ , it will be sufficient, for the time evolution up to $t = \tau$, to determine the functions $\Gamma_\tau^{(n)}$ and thus the propagator

$$G_{\tau,ab} = i[\Gamma_\tau^{(2)} + R_\tau]_{ab}^{-1}. \quad (8)$$

We restrict ourselves to the case $\phi_a \equiv 0$, such that the action (7) implies that $\Gamma_\tau^{(3)} \equiv 0$, and thus the flow of $\Gamma_\tau^{(1)}$ vanishes. Moreover, the equation for the proper two-point function involves, on the right hand side, only the term containing $\Gamma_\tau^{(4)}$,

$$\partial_\tau \Gamma_{\tau,ab}^{(2)} = \frac{i}{2} \int_C \Gamma_{\tau,abcd}^{(4)} (G_\tau [\partial_\tau R_\tau] G_\tau)_{dc}, \quad (9)$$

see also Fig. 3. The term in parentheses stands for the regularised line. We supplement Eq. (9) with the flow equation for $\Gamma_\tau^{(4)}$, which, for $\phi_a \equiv 0$, is depicted in Fig. 3. This system of equations is still exact. For practical computations it needs to be closed which can be achieved by truncation or by supplementing it with equations for one or more higher n -vertices truncated at some higher order. Here we truncate by neglecting, in the equation for $\Gamma_\tau^{(4)}$, the term involving $\Gamma_\tau^{(6)}$,

$$\begin{aligned} \partial_\tau \Gamma_{\tau,abcd}^{(4)} = & -\frac{1}{8} \int_C \left\{ \Gamma_{\tau,abef}^{(4)} G_{\tau,fg} \Gamma_{\tau,cdgh}^{(4)} \right\} \\ & \times (G_\tau [\partial_\tau R_\tau] G_\tau)_{he} + P(a, b, c, d). \end{aligned} \quad (10)$$

P implies a sum corresponding to all permutations of its arguments. In this way we obtain a closed set of integro-differential equations for the proper functions up to fourth order. As we will show in the following, they allow to derive, for a particular cutoff time τ , a set of dynamic equations describing the time evolution of the two- and four-point functions up to time $t = \tau$. We emphasise that the only approximation here is the neglect of the six-point vertex, see Fig. 3.

Dynamic equations. For the sharp temporal cutoff R_τ chosen here the flow equations can be analytically integrated over τ . As pointed out above, our cutoff implies the connected two-point function to vanish at times greater than τ , i.e., it can be written as

$$G_{\tau,ab} = i \left[\Gamma_\tau^{(2)} \right]_{ab}^{-1} \theta(\tau - t_a) \theta(\tau - t_b), \quad (11)$$

where $\theta(\tau)$ evaluates to 0 for $\tau < 0$ and to 1 elsewhere, and where t_a is the time argument corresponding to the field index a , etc. Hence, the precise way in which the cutoff $R_{\tau,ab}$ diverges at $t_a = t_b > \tau$ is chosen such that $\Gamma_\tau^{(2)} + R_\tau$ is the inverse of $-iG_\tau$ for all times t_a, t_b , see Eq. (8). Using Eq. (11) one finds that

$$(G_\tau [\partial_\tau R_\tau] G_\tau)_{ab} = -i G_{\tau,ab} \partial_\tau [\theta(\tau - t_a) \theta(\tau - t_b)]. \quad (12)$$

Note that we have not used the specific choice (4) for deriving (12) but simply the property (11), that is the suppression of any propagation for times $t > \tau$. This independence of the specific choice of R_τ leading to (11) is the generic feature of the present approach. After inserting Eq. (12) into Eqs. (9) and (10), we can integrate over τ and obtain, after some algebra, the integral equations determining the flow of the proper

functions from t_0 to some final time t ,

$$\Gamma_{\tau,ab}^{(2)} \Big|_{t_0}^t = \frac{1}{2} \int_{t_0, \mathcal{C}}^t \Gamma_{\tau_{cd},acbd}^{(4)} G_{\tau_{cd},dc}, \quad (13)$$

$$\begin{aligned} \Gamma_{\tau,abcd}^{(4)} \Big|_{t_0}^t &= \frac{i}{2} \int_{t_0, \mathcal{C}}^t \Gamma_{\tau_{efgh},abef}^{(4)} G_{\tau_{fg},fg} \\ &\times \Gamma_{\tau_{efgh},cdgh} G_{\tau_{eh},he} + (a \leftrightarrow c) + (a \leftrightarrow d). \end{aligned} \quad (14)$$

Double indices imply sums over field components, spatial integrals and time integrations over the CTP \mathcal{C} , from t_0 to t and back to t_0 . We furthermore introduced

$$\begin{aligned} \tau_{ab} &= \max\{t_a, t_b\}, \\ \tau_{abcd} &= \max\{t_a, t_b, t_c, t_d\}. \end{aligned} \quad (15)$$

The brackets denote terms with the respective indices swapped.

From Eqs. (13) and (14) it is clear that for the two- and four-point functions to be defined at $\tau = t$ we need to specify initial functions at $\tau = t_0$. We point out that, within the truncation scheme chosen above, we can insert any set of proper two- and four-point functions defined in their time arguments at and only at t_0 , as long as we set all n -vertices for $n = 1, 3$, and $n > 4$ to vanish. Our scheme corresponds to a Gaussian initial density matrix $\rho_D(t_0)$ since the four-point function has an influence on $\rho_D(t)$ only for $t > t_0$. Here, we choose the respective classical proper functions defined by S in Eq. (7). Hence, the initial two- and four-point functions entering Eqs. (13) and (14) read

$$\Gamma_{t_0,ab}^{(2)} = S_{ab}^{(2)} = iG_{0,ab}^{-1}, \quad (16)$$

$$\begin{aligned} \Gamma_{t_0,abcd}^{(4)} &= S_{abcd}^{(4)} = -(2g/\mathcal{N})(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \\ &\times \delta_{\mathcal{C}}(x_a - x_b)\delta_{\mathcal{C}}(x_b - x_c)\delta_{\mathcal{C}}(x_c - x_d). \end{aligned} \quad (17)$$

In order to arrive at a set of dynamic differential equations, we finally rewrite Eq. (13) as

$$iG_{0,ac}^{-1} G_{\tau_{cb},cb} = i\delta_{\mathcal{C},ab} - \frac{1}{2} \int_{t_0, \mathcal{C}}^{\tau_{cb}} \Gamma_{\tau_{de},adce}^{(4)} G_{\tau_{de},ed} G_{\tau_{cb},cb}, \quad (18)$$

with $\delta_{\mathcal{C},ab} = \delta_{ab}\delta_{\mathcal{C}}(x_a - x_b) = \delta_{ab}\delta_{\mathcal{C}}(t_a - t_b)\delta(\mathbf{x}_a - \mathbf{x}_b)$. Eq. (18) is the dynamic (Dyson-Schwinger) equation for the connected two-point function $G_{\tau_{ab},ab}$. Since $\Gamma_{t_0}^{(2)} = iG_{0}^{-1}$ represents a differential operator, the solution of Eq. (18) finally requires another set of boundary conditions to be specified, depending on the form of the differential operator. In our case, this is the initial two-point function, fixed by the one-body density matrices as well as Bose statistics [9]. We finally point out that we have set $t = \tau_{cb}$ since the flow of $G_{\tau_{cb},cb}$ stops at the maximum of the time arguments t_b, t_c . This can be proven from the structure of Eqs. (14), (18) but is more

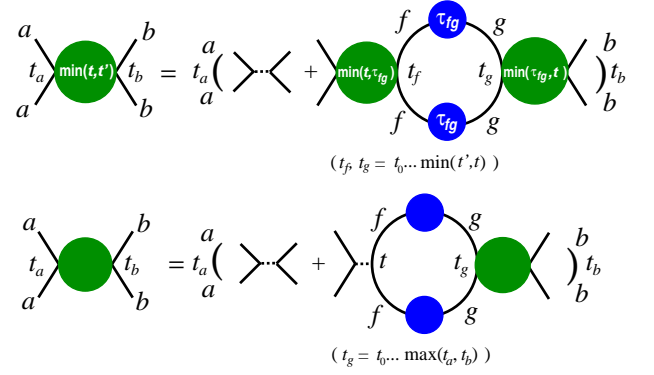


FIG. 4: (color online) The upper equation is the s -channel projection of Eq. (14). The second equation defines the resummed vertex appearing in the Dyson-Schwinger equation derived from the NLO $1/\mathcal{N}$ approximation of the 2PI effective action [5]. The two definitions are identical in every order of a perturbative expansion (see text). Dashed lines denote the s -channel part of the bare vertex $\Gamma_{t_0,ab}^s$, see text. All other symbols correspond to those in Fig. 3. Letters on internal lines indicate summation over field indices and integration over space and time (along the CTP from t_0 to t and back). The integration intervals are given in parentheses.

easily seen from the definition (3): Once the hard cutoff τ has passed the largest time appearing in a (connected) correlation function, the flow stops since the forward and backward parts of the CTP over greater times cancel identically in the functional integral.

Let us assume that $t = t_a$ denotes the present time, at which Eq. (18) determines the further propagation of $G_{t_a,ab}$ for $t_b \leq t_a$ (for $t_b > t_a$ the solution is then fixed by symmetry). We point out that all time arguments of the functions occurring on the right-hand sides of Eqs. (13), (14), and therefore all cutoff times τ_{ab} and τ_{efgh} are evaluated at times $t' \leq t$. Hence, in accordance with causality, Eqs. (14) and (18) for a given initial $G_{t_0,ab}(t_0, t_0)$, can be solved iteratively.

This concludes the derivation of our main results, a closed set of dynamic equations for the two-point correlation function as obtained from a functional RG approach with a cutoff in real time. In the remainder of this article we will concentrate on rederiving the intensively used dynamic equations obtained from the 2PI effective action in NLO of an expansion in inverse powers of the number of field components \mathcal{N} and comment on the potential of our approach beyond this approximation.

From RG to 2PI next-to-leading order (NLO) $1/\mathcal{N}$. The mean-field approximation, i.e., the well-known Hartree-Fock equations are obtained by neglecting the flow of the four-point according to Eq. (14), $\Gamma_t^{(4)} \equiv \Gamma_{t_0}^{(4)}$. As a consequence, the flow parameter of $G_{\tau,ab}$ is, for the chosen cutoff, fixed to $\tau = \tau_{ab}$ and can thus be neglected. Cf. Ref. [9] for the mean-field equations in the notation used here.

As a first step beyond mean-field we consider the truncation in which the s -channel scattering diagram is included beyond the mean-field limit in the loop integral on the right-hand side of Eq. (14). The result of this section will be

that the obtained equations are equivalent to the 2PI equations in NLO of a $1/\mathcal{N}$ expansion [5]. This truncation corresponds to keeping only one channel of each, the classical vertex and the one-loop integral term. The four-vertex then only depends on two space-time variables and field indices, $\Gamma_{t,acbd}^{(4)} = \Gamma_{t,ab}^{(4)s} \delta_{ac} \delta_{bd}$, and enters, in Eq. (18), as a cutoff-dependent self energy $\Sigma_{\tau_{ab},ab} \equiv \Gamma_{\tau_{ab},ab}^{(4)s} G_{\tau_{ab},ab}$. We can always write $t = \min(t, t')$, with $t = t' = \tau_{ab}$, for the cutoff parameter in $\Gamma_{t,ab}^{(4)s}$, see Fig. 4, upper line. Since the parameter in the (green) vertices on the right hand side is also the minimum of the maxima of integration times in the adjacent loop and the respective external time t or t' , one can iterate the integral equation in order to obtain a perturbative series of bubble-chain diagrams consisting only of classical vertices and full, cutoff-dependent propagators. This procedure provides us with a proof that $\Sigma_{\tau_{ab},ab}$ is *identical* to the NLO 2PI $1/\mathcal{N}$ self energy obtained from the 2PI effective action, cf. Ref. [5] and Fig. 4, lower line.

This proof which exploits the perturbative expansion is lengthy and will be given elsewhere. Here we show that the above identity can be inferred in a comparatively easy way from the topology of the different terms in the flow equations for the two-, four- and six-point functions: Consider the untruncated set of equations as displayed in Fig. 3. First, non- s -channel contributions do not generate bubble-chains of the form shown in Fig. 4. Second, $\Gamma_{\tau}^{(6)}$ is one-particle irreducible. Its contribution to the flow of $\Gamma_{\tau}^{(4)}$ does not give rise to bubble-chains, even if inserted recursively into the first diagram on the right-hand side of the flow equation for $\Gamma_{\tau}^{(4)}$. In turn, by dropping the second diagram with the six-point function and using the s -channel truncation, the iterated flow equation generates only bubble-chain diagrams with full propagators as lines. Hence, a τ -integration of this set of flow equations leads to dynamic equations which include all bubble-chain contributions and are therefore equal to those obtained to NLO in a $1/\mathcal{N}$ expansion of the 2PI effective action [5]. We emphasise that the above topological arguments are generally valid when comparing resummation schemes inherent in RG equations of the type of Eq. (6), with those obtained from 2PI effective actions. This applies, e.g., to equilibrium flows [14, 15, 16] and thermal flows [15, 22]. For a comparison with 2PI results see Ref. [22], for the interrelation of 2PI methods and RG flows Ref. [16]. In this context we would also like to remark that our approach, by constructions, obeys Goldstone's theorem as it is solely based on the 1PI effective action and its derivatives, the proper n -point functions. Hence, spontaneous symmetry breaking always goes together with gapless excitations in the proper 2-point functions. For a resolution of this matter in the 2PI approach see e.g. Refs. [6, 23].

Equilibration of a strongly interacting Bose gas. We close our paper with an illustration of the non-perturbative nature of the dynamic equations obtained in the s -channel approximation, which are, as shown, equivalent to the 2PI equations in NLO $1/\mathcal{N}$ approximation. For this we face the long-time evolution of a weakly interacting non-relativistic Bose

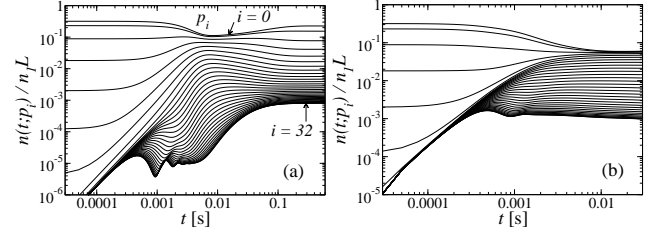


FIG. 5: Time evolution of momentum-mode occupation numbers $n(t; p)/n_1 L$ of a uniform 1D Bose gas, normalised by the total number $n_1 L$ of atoms in the box. The initial far-from-equilibrium state corresponds to a Gaussian momentum distribution. Panel (a) describes the evolution of a weakly interacting gas, $\gamma = 1.5 \cdot 10^{-3}$, panel (b) that of a strongly interacting gas with $\gamma = 15$.

gas obtained in Ref. [11], for which $1/\mathcal{N} = 1/2$, with that of a strongly interacting one. These results illustrate, what is supported by the non-perturbative approach presented here as well as by benchmark tests for special-case systems [24], viz. that the s -channel or 2PI $1/\mathcal{N}$ approximation is applicable for strong coupling also for small \mathcal{N} where an expansion in powers of $1/\mathcal{N}$ would seem questionable. The uniform gas in one spatial dimension starts from a non-equilibrium state which is assumed to be fully described by a non-vanishing two-point function $G_{aa}(t_0, t_0; p) = n(t_0; p) + 1/2$, $a = 1, 2$. The initial momentum distribution of atoms in the gas is $n(t_0; p) = (n_1 \sqrt{\pi} \sigma) \exp(-p^2/\sigma^2)$, with $\sigma = 1.3 \cdot 10^5 \text{ m}^{-1}$. The computations were done as described in Ref. [11], on a grid with $N_s = 64$ points spaced by $a_s = 1.33 \mu\text{m}$. Fig. 5 shows the evolution for a (a) weakly interacting sodium gas, with line density $n_1 = 10^7$ atoms/m and interaction strength $g = \hbar^2 \gamma n_1 / m$, $\gamma = 1.5 \times 10^{-3}$ and (b) a strongly interacting one with $\gamma = 15$, $n_1 = 10^5/\text{m}$. The final momentum distributions closely resemble Bose-Einstein, i.e., thermal distributions $n(t; p) \propto 1/[\exp(\beta[\omega_p - \mu]) - 1]$, with the dispersion ω_p extracted from the numerical data [11].

Conclusions. Using functional RG techniques we have derived a theory of far-from-equilibrium quantum field dynamics. The time evolution of the system is generated through the flow with a cutoff time introduced in the generating functional for correlation functions. In a truncation of the flow equations we recover the dynamic equations as known from the $1/\mathcal{N}$ -expansion of the 2PI effective action. These results lead us to the following conclusions: Although sub-leading terms in the expansion of the 2PI effective action are suppressed with additional powers of $1/\mathcal{N}$ they contain bare couplings, and, despite their resummation, the approximation seems formally questionable for strong couplings if g/\mathcal{N} is large. Presently, an intense discussion focuses on the question to what extent the NLO $1/\mathcal{N}$ approximation is applicable for large interaction strengths, drawing from benchmark tests for special-case systems [24]. Our results provide analytical means to find the conditions for the validity of this approximation since no perturbative ordering in bare couplings is involved, and the remaining truncation in the Green functions can be tested for self-consistency.

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* email: T.Gasenzer@thphys.uni-heidelberg.de

† email: J.Pawlowski@thphys.uni-heidelberg.de

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